

Problem 10406 from The American Mathematical Monthly, Volume 101, Number 8 / October 1994

10406. Proposed by David C. Fisher, University of Colorado, Denver, CO, Karen L. Collins, Wesleyan University, Middleton, CT, and Lucia B. Krompart, Rochester, MI.

Show that a path on an m by n square grid which starts at the northwest corner, goes through each point exactly once, and ends at the southeast corner divides the grid into two equal halves: (a) those regions opening north or east; and (b) those regions opening south or west.

(10406) A path meeting the conditions of the problem on a 5 by 8 grid is shown in figure 10406 below.

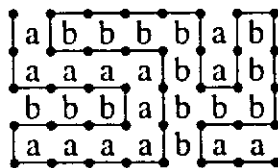
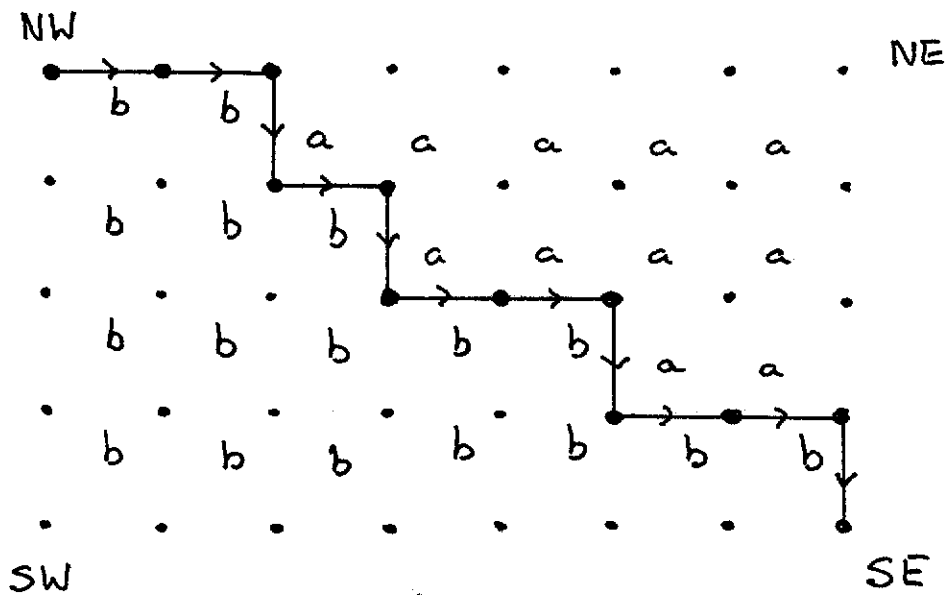


Figure 10406

We confine our attention to paths from NW to SE that do not necessarily go through each point; the off-path points are called free. Directing the paths from NW to SE, I identify (here without proof) the region(s) "a" with those to the left of the path, and the regions "b" with those to its right.

A path, along which turns to the left and turns to the right alternate, is a shortest path; we give a simple example. (in the same 5 by 8 grid).



For the m by n grid with a shortest path we observe

- # points = $m * n$
- # unit squares = $(m-1) * (n-1)$
- # edges on shortest path = $(m-1) + (n-1)$
- # points on shortest path = $m + n - 1$
- # free points = $(m * n) - (m + n - 1)$
 $= (m-1) * (n-1)$

in short: the number of unit squares equals the number of free points. But we can go further: this equality holds for the separate regions! We can establish in the a -region a 1-1 correspondence between each unit square and its NE corner, and in the b -region between each unit square and its SW corner.

Introducing

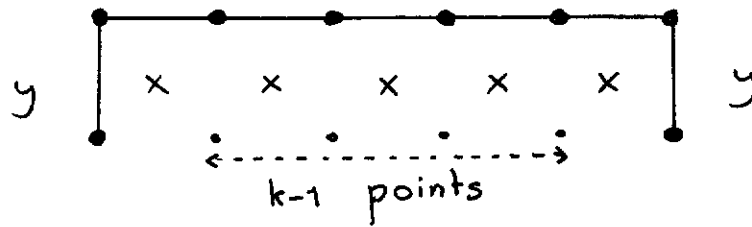
F_a and F_b : # free points in a -region(s)

and b -region(s) respectively
 S_a and S_b : # unit squares in a -region(s)
 and b -region(s) respectively,
 we can now formulate

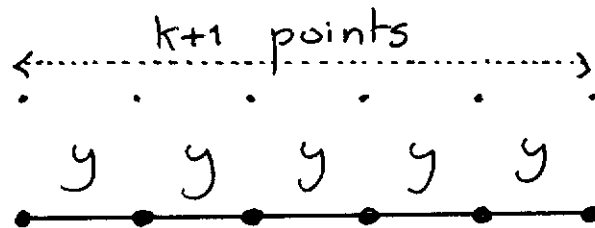
Lemma 0 In the case of a shortest path
 $F_a = S_a \wedge F_b = S_b$.

The idea of the proof is to start with a given path that visits each point exactly once - i.e. $F_a = 0 \wedge F_b = 0$ - , and to transform it in a finite number of moves into a shortest path for which $F_a = S_a \wedge F_b = S_b$ holds. Initial and final state are then to be linked via an invariant that is maintained by each move.

For the move we consider that, as said, a path, along which turns to the left and turns to the right alternate, is a shortest path. Consequently, if the path is not of minimal length, it contains at least one pair of successive turns in the same direction. With those turns k edges apart it means that there exists a rectangle - a "bar" - of k ($k \geq 1$) unit squares such that 3 of its sides are formed by 1, k , 1 edges of the path, e.g. for

$k = 5:$ 

We can -and will- always choose a rectangle such that the $k-1$ interior points of the 4th side are free. The move shortens the path by 2 edges in the obvious way; in the same example, the above configuration is transformed into



With $x, y = a, b$ or $x, y = b, a$, we observe for the transformation by the move:

$$\Delta S_x = -k$$

$$\Delta F_x = -(k-1)$$

$$\Delta(S_x - F_x) = -1$$

$$\Delta S_y = +k$$

$$\Delta F_y = +k+1$$

$$\Delta(S_y - F_y) = -1$$

The above transformation reduces at both sides of the path the difference $S-F$ by 1, and hence

$$(S_a - F_a) - (S_b - F_b) = q$$

if initially true, is an invariant of our total transformation process (which ob-

viously terminates since each move reduces the path by 2 edges).

From Lemma 0, which is applicable in the final state, we conclude $q = 0$. Therefore initially, with $F_a = 0 \wedge F_b = 0$, we have $S_a - S_b = 0$ or $S_a = S_b$. Quod erat demonstrandum.

* * *

It took me several hours to design the move of the previous page; for quite a while I considered 2 unparameterized moves: the current one with $k=1$ and " $\Gamma \rightarrow \Gamma'$ ", which changes the shape of the path without shortening it. The mere fact that the second move complicates the termination argument should have made me reject it much faster.

I liked the problem.

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prof. dr. Edsger W. Dijkstra
 Department of Computer Sciences
 The University of Texas at Austin
 Austin, TX 78712-1188
 USA