A more disentangled characterization of extreme solutions

Let $\sqsubseteq$ —read "below"— be a punctual partial order. Let equation $x: [B.x]$ have a lowest solution k. The traditional characterization of k has been that it satisfies

(0) $[B.k]$

(1) $\langle \forall x : [B.x] : [k \sqsubseteq x] \rangle$,

where (0) expresses that k is a solution, and (1) states that k is everywhere below any solution.

Here I propose to characterize k by

(2) $\langle \exists x : [B.x] : [x \sqsubseteq k] \rangle$

(3) $\langle \forall x : [B.x] : [k \sqsubseteq x] \rangle$.

(Here, (3) is a copy of (1): I have repeated the formula so as to ease its comparison with (2).) We have to prove the

<u>Theorem</u> $(0) \wedge (1) \equiv (2) \wedge (3)$.

<u>Proof</u> For ping it suffices to show $(0) \Rightarrow (2)$:

$\langle \exists x : [B.x] : [x \sqsubseteq k] \rangle$

$\Leftarrow$ {instantiation $x := k$}

$$[B.k] \wedge [k \sqsubseteq k]$$
$= \quad \{\sqsubseteq$ is a partial order, hence reflexive$\}$
$$[B.k]$$

For pong it suffices to show $(0) \Leftarrow (2) \wedge (3)$:

$$\langle \exists x : [B.x] : [x \sqsubseteq k] \rangle \wedge \langle \forall x : [B.x] : [k \sqsubseteq x] \rangle$$
$\Rightarrow \quad \{$pred. calc.$\}$
$$\langle \exists x : [B.x] : [x \sqsubseteq k] \wedge [k \sqsubseteq x] \rangle$$
$\Rightarrow \quad \{\sqsubseteq$ is a partial order, hence antisymmetric$\}$
$$\langle \exists x : [B.x] : [x = k] \rangle$$
$= \quad \{$trading$\}$
$$\langle \exists x : [x = k] : [B.x] \rangle$$
$= \quad \{$1-point rule$\}$
$$[B.k]$$

(End of Proof)

The reason why (2)&(3) is such a nice pair is that (2) – monotonic in k – <u>only</u> bounds k from below, while (3) – antimonotonic in k – bounds k <u>only</u> from above. In the pair (0)&(1), (1) is obviously as nice as (3), but (0) – being in general <u>not</u> monotonic in k – is <u>not</u> a one-sided constraint. (It is not amazing that in the above pong argument we needed in the antecedent (3) as well.)

As an example of the heuristic guidance provided by this disentanglement we shall prove the following, now trivial, theorem.

<u>Theorem</u> Let k be the lowest solution of $x: [B.x]$; let h be the lowest solution of $x: [C.x]$. Then

$$\langle \forall x :: [C.x] \Rightarrow [B.x] \rangle \quad \Rightarrow \quad [k \sqsubseteq h] \quad .$$

<u>Proof</u> Analogously to (2)&(3), h is defined by

(4) $\langle \exists x : [C.x] : [x \sqsubseteq h] \rangle$

(5) $\langle \forall x : [C.x] : [h \sqsubseteq x] \rangle$.

In view of the demonstrandum, (3) gives <u>all</u> we need to know about k and (4) gives <u>all</u> we need to know about h.

$\langle \forall x :: [C.x] \Rightarrow [B.x] \rangle$
$\Rightarrow$ { (3) ; $\forall$ antimonotic with respect to range }
$\langle \forall x : [C.x] : [k \sqsubseteq x] \rangle$
$\Rightarrow$ { (4) , predicate calculus }
$\langle \exists x : [C.x] : [k \sqsubseteq x] \wedge [x \sqsubseteq h] \rangle$
$\Rightarrow$ { $\sqsubseteq$ is a partial order, hence transitive }
$\langle \exists x : [C.x] : [k \sqsubseteq h] \rangle$
$\Rightarrow$ { predicate calculus }
$[k \sqsubseteq h]$

(End of Proof.)

The above heuristics are tighter than those for the proof that for monotonic f , g is monotonic when $g.x$ is given as the strongest solution of $y: [f.x.y \Rightarrow y]$ — see Dijkstra, Scholten, 90, p. 153 — .

Similarly, the highest solution h of $x: [B.x]$ is given by

(6) $\langle \exists x: [B.x] : [h \sqsubseteq x] \rangle$

(7) $\langle \forall x: [B.x] : [x \sqsubseteq h] \rangle$.

<u>Remark</u> In the classical notation, which does not have a range, these formulae would not be half as nice. (End of Remark.)

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I am utterly amazed that it took me more than a decade to come up with the above characterization of extreme solutions.

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