On a formula designed by Henk Doornbos

We are asked to prove for any x, y, z

(0) $[x;y \wedge z \Rightarrow (z;\sim y \wedge x);(\sim x;z \wedge y)]$.

We don't know beforehand how much of the relational calculus we shall need, but it is almost certain that we shall need the exchange rules

$$[x;y \Rightarrow z] \equiv [\neg z;\sim y \Rightarrow \neg x]$$
$$[x;y \Rightarrow z] \equiv [\sim x;\neg z \Rightarrow \neg y] \qquad ,$$

as they embody the axiom connecting the transposition "$\sim$" and the composition "$;$" . In a moment we shall need that composition is monotonic; we may need that composition is universally disjunctive (from which the monotonicity follows).

In (0) , it is the consequent that needs to be unravelled, because nothing nice is known about the compositions of conjunctions. So we replace the consequent by $r;s$ and since

(1) $[x;y \wedge z \Rightarrow r;s]$

is a monotonic function of r and s , we can apply the theorem of EWD1117 .

Because, in view of the exchange rule, the expressions replaced are quite acceptable as antecedents, we use from EWD1117 that for monotonic f

$$[f.exp] \equiv \langle \forall z : [exp \Rightarrow z] : [f.z] \rangle \quad ,$$

i.e. we propose to prove (0) by showing that for any r, s , (1) follows from

(2a) $\quad [z; \sim y \wedge x \Rightarrow r]$

(2b) $\quad [\sim x; z \wedge y \Rightarrow s]$.

Because z occurs fairly isolated in (1) we try to isolate z from (2). We observe

$$
\begin{array}{ll}
& (2) \\
= & \{\text{definition}\} \\
& [z; \sim y \wedge x \Rightarrow r] \wedge [\sim x; z \wedge y \Rightarrow s] \\
= & \{\text{shunting in preparation for exchanges}\} \\
& [z; \sim y \Rightarrow \neg x \vee r] \wedge [\sim x; z \Rightarrow \neg y \vee s] \\
= & \{\text{exchange rules and de Morgan}\} \\
& [(x \wedge \neg r); y \Rightarrow \neg z] \wedge [x; (y \wedge \neg s) \Rightarrow \neg z] \\
= & \{\text{predicate calculus}\} \\
& [(x \wedge \neg r); y \vee x; (y \wedge \neg s) \Rightarrow \neg z]
\end{array}
$$

Using the above rewriting of (2) we observe

$$
\begin{array}{ll}
& (1) \\
= & \{\text{definition}\}
\end{array}
$$

$$
\begin{array}{ll}
 & [x;y \wedge z \Rightarrow r;s] \\
= & \{\text{shunting}\} \\
 & [x;y \Rightarrow r;s \vee \neg z] \\
\Leftarrow & \{(2)\text{ as rewritten}\} \\
 & [x;y \Rightarrow r;s \vee (x \wedge \neg r);y \vee x;(y \wedge \neg s)] \\
= & \{\text{see below}\} \\
 & true
\end{array}
$$

Our final obligation is to weaken $x;y$ to the above disjunction, at the same time introducing r, s, and the conjuncts $\neg r, \neg s$. We observe to this end

$$
\begin{array}{ll}
 & x;y \\
= & \{[y \equiv (y \wedge s) \vee (y \wedge \neg s)],\ \text{";" is disjunctive}\} \\
 & x;(y \wedge s) \vee x;(y \wedge \neg s) \\
= & \{[x \equiv (x \wedge r) \vee (x \wedge \neg r)],\ \text{";" is disjunctive}\} \\
 & (x \wedge r);(y \wedge s) \vee (x \wedge \neg r);(y \wedge s) \vee x;(y \wedge \neg s) \\
\Rightarrow & \{\text{";" is monotonic}\} \\
 & r;s \vee (x \wedge \neg r);y \vee x;(y \wedge \neg s)
\end{array}
$$

QED

Remark I met the formula in the form (1) $\Leftarrow$ (2), in which form it was attributed to Henk Doornbos (and called the Grand Dedekind rule).

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