A much neglected mathematical object

I had noticed that, for many years, about my only use of the notion of a function was via the Rule of Leibniz, viz. that we have for any f, x, y of appropriately matching types

(0) $\quad x = y \Rightarrow f.x = f.y$.

I also knew that the more formal expression of Leibniz's Latin phrase is probably

(1) $\quad x = y \equiv (\underline{A}f{::}\ f.x = f.y)$,

but did not feel that that gave me much more: $\Rightarrow$ follows from (0) and $\Leftarrow$ follows from the existence of the identity function I .

A few months ago I realized that equality of functions is usually expressed by stating that we have for any g, h, x of appropriately matching types

(2) $\quad g = h \equiv (\underline{A}x{::}\ g.x = h.x)$.

I could not help being struck by the symmetry between (1) and (2) in function and argument; it made me wonder to what extent the traditional distinction between function and argument is primarily a linguistic artefact. I was genuinely puzzled as no effort of mine at being more explicit about the appropriately matching types succeeded in destroying the symmetry. In analogy to (0) I wrote down

(3) $\quad g = h \Rightarrow g.x = h.x \quad ,$

but this only underlines the symmetry: (0) and (3) present function application as <u>equality-preserving</u> in both respects.

Seeing how (1) follows from (0), we can deduce (2) from (3) with the aid of the so far neglected mathematical object: the identity argument i.

The introduction of the identity argument i indeed fully restores the symmetry between function and argument, but at the price mathematicians are traditionally unwilling to pay: the resulting theory only admits the trivial (i.e. one-point) model. The following argument is essentially due to Samson Abramsky. We have

(4) $\quad I.x = x \quad$ for any x

(5) $\quad f.i = f \quad$ for any f.

Define function F by

(6) $\quad F.x = I \quad .$

Then we observe for any x

x

$= \quad \{(4)\}$

$I.x$

$$
\begin{aligned}
= &\quad \{(6) \text{ with } x := i\} \\
& (F.i).x \\
= &\quad \{(5) \text{ with } f := F\} \\
& F.x \\
= &\quad \{(6)\} \\
& I \qquad ,
\end{aligned}
$$

hence $(\underline{A}x :: x = I)$ and $(\underline{A}x, y :: x = y)$.

The λ-calculus allows us to define —like in (4) and (6)— a function F by

F.x = an expression that may contain x

(if not, as in (6), we define what is known in the jargon as "a constant function").

The alternative is the permission to define —like in (5)— an argument X by

f.X = an expression that may contain f

(if not, we define what the jargon should call "a constant argument").

The combination of both freedoms is more than we care to live with.

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