A notational alternative for quantification.

In the following I restrict myself as usual to (beginnings of) countable domains; booleans and (enough) integers will do.

The so-called universal quantification is given by:

$$(\underline{A}\ i\colon 0 \le i < 0\colon B(i)) = \text{true}$$
$$(\underline{A}\ i\colon 0 \le i < n{+}1\colon B(i)) = (\underline{A}\ i\colon 0 \le i < n\colon B(i))\ \underline{\text{and}}\ B(n) \quad .$$

The so-called existential quantification is given by:

$$(\underline{E}\ i\colon 0 \le i < 0\colon B(i)) = \text{false}$$
$$(\underline{E}\ i\colon 0 \le i < n{+}1\colon B(i)) = (\underline{E}\ i\colon 0 \le i < n\colon B(i))\ \underline{\text{or}}\ B(n) \quad .$$

Augustus de Morgan's Law

$$\underline{\text{non}}\ (\underline{E}\ i\colon 0 \le i < n\colon B(i)) = (\underline{A}\ i\colon 0 \le i < n\colon \underline{\text{non}}\ B(i))$$

is easily proved by mathematical induction.

* * *

Quite a few years ago the above notations inspired another one, which I found under circumstances indispensable. It denotes a natural number, viz. the number of different values of i in a range such that B(i) holds. Its recursive definition is

$$(\underline{N}\ i\colon 0 \le i < 0\colon B(i)) = 0$$
$$\begin{aligned}(\underline{N}\ i\colon 0 \le i < n{+}1\colon B(i)) = \ &\underline{\text{if}}\ \underline{\text{non}}\ B(n) \rightarrow (\underline{N}\ i\colon 0 \le i < n\colon B(i))\\ &\ [\!]\ B(n) \rightarrow (\underline{N}\ i\colon 0 \le i < n\colon B(i)) + 1\\ &\underline{\text{fi}} \quad .\end{aligned}$$

Let us call it "the quantified count". Once the quantified count has been introduced, we don't need the universal and existential quantifiers anymore. They can be defined in terms of the quantified count:

$$(\underline{A}\ i\colon 0 \le i < n\colon B(i)) = ((\underline{N}\ i\colon 0 \le i < n\colon \underline{\text{non}}\ B(i)) = 0)$$
$$(\underline{E}\ i\colon 0 \le i < n\colon B(i)) = ((\underline{N}\ i\colon 0 \le i < n\colon B(i)) \ge 1) \quad .$$

We leave the proof that our new definitions are equivalent to our earlier ones as an exercise for the interested reader.

One advantage of the new notation is that Augustus de Morgan's Law becomes extremely simple. Omitting for shortness sake the ranges --and indicating this omission by a succession of two colons-- it becomes

$$(\underline{\text{non}}\ (\underline{N}\ i{::}\ B(i)) \geq 1) = ((\underline{N}\ i{::}\ B(i)) = 0) \quad ,$$

a relation that follows directly from the fact that a quantified count is a natural number.

All this was discovered quite a few years ago; I remember that it raised a mild suspicion that the introduction of two quantifiers where one quantified count would have done had perhaps been a notational mistake. In the meantime I encountered some more food for that suspicion.

*   *   *

The introduction of the universal and existential quantifiers reflects in two ways a doubtful practice. The minor complaint is that it is wasteful to introduce two special characters --in print they appear as inverted, bold, sans-serif A and E , respectively-- when, in addition, we then need Augustus de Morgan's Law (!) to remind us that the one is only the negation of the other.

(In this connection it might be worthwhile to point out that in the boolean domain the "operation" of negation may be regarded as trivial in the sense that you cannot determine a boolean value x without determining the value of non x as well. In the integer domain you cannot determine a value x by determining a value y such that $x \neq y$; in the boolean domain you can!)

I called this a minor complaint: for the sake of brevity some redundancy in special characters may be defended. (This admission is not to be construed as a carte blanche from my side for the notational immodesty that some of our less wise mathematicians indulge in.) The major complaint is the introduction of a special notation for what seems at first sight an arbitrary dichotomy. Is for natural numbers the dichotomy in "equal to zero" versus "at least one" really the only meaningful one?

Of course it isn't. In the mathematical literature one finds the existential quantifier sometimes adorned with an exclamation mark to indicate the existence of exactly one value of i such that B(i), i.e.

$$(\underline{E}!\ i{::}\ B(i)) = ((\underline{N}\ i{::}\ B(i)) = 1) \quad .$$

I do not know who introduced that exclamation mark --no doubt his name can be found in Florian Cajori's "A History of Mathematical Notations"-- ; the introduction of such specific notational gadgetry is of course a road sign saying "Blind Alley".

Of this I made a mental note when I decided to let the matter rest until I had encountered the need for more other dichotomies. Yesterday's experience prompted the writing of this note. During my lectures, when applying the Gries-Owicki Theory to a solution to the mutual exclusion problem between two processes, I derived the invariant

$$\underline{\text{non}}\ \text{luck}(0)\ \underline{\text{or}}\ \underline{\text{non}}\ \text{luck}(1) \qquad (x)$$

or, in the notation used above

$$(\underline{E}\ i{:}\ 0 \leq i < 2{:}\ \underline{\text{non}}\ \text{luck}(i)) \quad . \qquad (y)$$

When my lectures were over and I walked back to my room I realized that the analogous relation after generalization from 2 to n processes would not be

$$(\underline{E}\ i{:}\ 0 \leq i < n{:}\ \underline{\text{non}}\ \text{luck}(i))$$

but

$$(\underline{N}\ i{:}\ 0 \leq i < n{:}\ \text{luck}(i)) \leq 1 \quad ,$$

and that is a formula of a very different nature! In retrospect, formula (x), which suggests formula (y) so strongly, should be regarded as a notational trap (into which I duly fell).

Finally I ask my readers to savour the expression --in which the ranges have again been omitted--

$$(\underline{N}\ x{::}\ (\underline{N}\ i{::}\ P[i] = x) \neq (\underline{N}\ j{::}\ Q[j] = x)) = 0 \quad ,$$

expressing that the finite sequence represented by array P is a permutation of the one similarly represented by array Q . Rendered in English it would be something like "No value occurs different numbers of times in P and Q , respectively.".

Without thus paraphrasing it I gave the above formula to a few mathematicians to stare at; they all knew the notation of the quantified count --I had used it before-- but were by an accident of history more familiar with the universal and existential quantifiers. One of them needed a surprisingly long time to come to grips with it: one of his problems seemed to be to realize that a value occurring in neither of the two sequences occurs in both an equal number of times, viz. zero. The explanation may be that he had his original training in the Department of Mathematics in Nymegen, and there --I recently learned-- the natural numbers still start at 1 !

<u>Note</u>. There are two reasons why it is stupid to define the natural numbers as $\{1, 2, 3, 4, ...\}$ instead of as $\{0, 1, 2, 3, ...\}$. Firstly, we already have a perfect name for the set $\{1, 2, 3, 4, ...\}$, viz. the set of positive integers. Secondly, by excluding zero from the natural numbers it becomes "unnatural", and the fact that for many the number "zero" is still a second-grade citizen is a standard stumbling block in mathematical thinking. In this case the situation is aggravated because, the University of Nymegen being a Roman Catholic one, the term "unnatural" is there loaded with more pejorative connotations than elsewhere. (End of Note.)

The writing of the above was a pleasant --and I think fully appropriate-- way of spending the day of the Abdication of Queen Juliana and the Coronation of Queen Beatrix. The weather is much nicer than predicted and I trust to become as grateful for her reign to Queen Beatrix as I am to who is now Princess Juliana again. The writing and subsequent typing of the above took me five hours; I consider them well-spent.

Plataanstraat 5
5671 AL NUENEN
The Netherlands

30 April 1980

prof.dr.Edsger W.Dijkstra
Burroughs Research Fellow