<u>A proof of a theorem communicated to us by S.Ghosh</u>.

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In a letter of 19 August 1976, S.Ghosh (currently c/o Lehrstuhl Informatik I, Universität Dortmund, Western Germany) communicated without proof the following theorem in natural numbers --here chosen to mean "nonnegative integers"-- :

Given a set of $k$ linear equations of the form

$$L_i = b_i \qquad (0 \le i < k) \tag{1}$$

in which the $L_i$ are homogeneous linear expressions in the unknowns with natural coefficients and the $b_i$ are natural numbers, there exists a single equation

$$M = c \tag{2}$$

in which $M$ is a homogeneous linear expression in the unknowns with natural coefficients and $c$ is a natural number, such that (2) has the same natural solutions as (1).

Because the natural solutions of (1) are the <u>common</u> natural solutions of (3) and (4), as given by

$$\begin{aligned} L_0 &= b_0 \\ L_1 &= b_1 \end{aligned} \tag{3}$$

and

$$L_i = b_i \quad \text{for} \quad 2 \le i < k \tag{4}$$

it suffices to prove that (3) can be replaced by a single equation with the same natural solutions as (3).

Consider for natural $p_0$ and $p_1$ , to be chosen later, the equation

$$p_0 * L_0 + p_1 * L_1 = p_0 * b_0 + p_1 * b_1 \tag{5}$$

All solutions of (3) are solutions of (5). We shall show that $p_0$ and $p_1$ can be chosen in such a way, that, conversely, all natural solutions of (5) are solutions of (3). We shall do so by choosing $p_0$ and $p_1$ in such a way that (5), considered as an equation in $L_0$ and $L_1$ , has (3) as its <u>only</u> natural solution; because all natural choices for the original unknowns will give rise to natural $L_0$ and $L_1$ , this is sufficient.

Considered as an equation in $L_0$ and $L_1$ , the general parametric solution of (5) is given by

$$L_0 = b_0 + t * p_1$$
$$L_1 = b_1 - t * p_0$$

(where, to start with, t need not be a natural number). We shall choose a natural $p_0$ and $p_1$ in such a way that from natural $L_0$ and $L_1$ , viz.

$$b_0 + t * p_1 \geq 0 \qquad (6)$$
$$b_1 - t * p_0 \geq 0 \qquad (7)$$
left-hand sides of (6) and (7) integer (8)

we can conclude $t = 0$ .

Choosing $p_1 > b_0$ , we derive from (6)

$$t > -1 \qquad (9)$$

Choosing $p_0 > b_1$ , we derive from (7)

$$t < 1 \qquad (10)$$

Choosing $p_0$ and $p_1$ furthermore such that $gcd(p_0, p_1) = 1$ , we derive from (8) that t must be integer; in view of (9) and (10) we conclude that $t = 0$ holds. Summarizing: (5) can replace (3) provided

$$p_0 > b_1 , \quad p_1 > b_0 , \quad gcd(p_0, p_1) = 1 \quad .$$

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Example. Let the given set be $x = 1, y = 1, z = 1$ . The first two equations can be combined by choosing $p_0 = 2$ and $p_1 = 3$, yielding:

$$2*x + 3*y = 5 , \quad z = 1 \quad .$$

These two can be combined by choosing $p_0 = 2$ and $p_1 = 7$, yielding

$$4*x + 6*y + 7*z = 17$$

for which (1,1,1) is, indeed, the only natural solution. (End of example.)

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